TREATMENT OF PULSATING WHITE DWARFS INCLUDING GENERAL RELATIVISTIC EFFECTS

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(Received 1 November 1968; revised 10 March 1969)

Abstract. In this paper, pulsating white dwarfs are treated via general relativity. Numerical integration of Einstein's equations was used to find equilibrium white dwarfs models and the fundamental periods of small oscillations about these equilibrium models. In these calculations account was taken of coulomb, Thomas-Fermi, and exchange interactions as well as ion zero point energies. It is shown that general relativity makes not just a quantitative difference in the results but a qualitative difference; pure C^{12} models which are stable in Newtonian mechanics can be unstable against collapse (at a central density of 3×10^{10} g/cm^3) when general relativity is taken into account. The collapsing model may become a neutron star or may continue towards the Schwarzschild radius.

More realistic white dwarf models with carbon burning products at the center, also were studied. For these models, the density at which the star becomes unstable against collapse due to electron capture (3×10^9 g/cm^3) was found to be lower than the density at which general relativistic instability occurs.

1. Introduction

The discovery of pulsars by Hewish *et al.* (1968) has generated renewed interest in pulsating stars. In particular, pulsating white dwarfs have been suggested as a possible candidate (Hewish *et al.*, 1968; Thorne and Ipser, 1968). Without suggesting any connection with pulsars, we present here some calculations of the fundamental periods of very dense white dwarf stars. Unlike Newtonian white dwarf models (Skilling, 1968a; Cocke and Cohen, 1968), general relativistic white dwarf models have a minimum fundamental period (Cohen, 1968; Faulkner and Gribbin, 1968; Skilling, 1968b) for a given composition.

Because of the destabilizing effect of general relativity on stellar models, white dwarf models which are stable in Newtonian mechanics may be dynamically unstable when general relativity is taken into account. In such models, general relativity gives results which are qualitatively different from those of Newtonian mechanics. This has previously been discussed by Chandrasekhar and Tooper (1964) and by Kaplan (1949, 1953). This is true for white dwarfs even though $2Gm/c^2r \sim 10^{-3}$. In Newtonian mechanics, one finds that the period decreases as the density increases until electron capture begins (Skilling, 1968a; Cocke and Cohen, 1968). This is of intrinsic interest in white dwarfs

Astrophysics and Space Science **5** (1969) 113–125; © *D. Reidel Publishing Company, Dordrecht-Holland*

together with the related parameters such as maximum mass (Chandrasekhar, 1935), central density, and period.

In the following sections, we present some results of a machine integration of Einstein's field equations. These integrations give the masses and radii of white dwarf models as functions of central density, as well as the periods of small radial pulsations about the equilibrium positions. These periods were obtained by integrating the eigenvalue equations describing the pulsations rather than via variational techniques. This method gives both accurate eigenfunctions and eigenvalues. Also, the results do not depend on any assumed polynomial forms for the eigenfunctions.

For the equilibrium model, Einstein's equations take the well known form (Landau and Lifshitz, 1962)

$$m_r = 4\pi r^2 \varrho \tag{1}$$

$$c^2 \varphi_r = \frac{G(m + 4\pi r^3 p/c^2)}{r(r - 2Gm/c^2)} \tag{2}$$

$$p_r = -(\varrho + p/c^2)\, \varphi_r c^2 \tag{3}$$

where the Schwarzschild radial coordinate r is defined (Cohen and Cohen, 1969) such that the surface area of a sphere of radius r is $4\pi r^2$, the subscript r denotes differentiation with respect to r, φ corresponds in the weak field limit to the gravitational potential, and m is the Schwarzschild mass.

Although degenerate electrons make the predominant contribution to the pressure p, coulomb, Thomas-Fermi, and exchange interactions (Hamada and Salpeter, 1961) were also considered since they lower the pulsation period slightly. Similarly, baryons make the main contribution to the energy density ϱ, but electron kinetic (and rest) energy, coulomb, Thomas-Fermi, and exchange energies were also taken into account. These contributions to the energy density must be included for consistency with the corresponding pressure; this has not been done in the past.

The pulsation equation has been obtained by a number of authors directly from Einstein's equations (Chandrasekhar, 1964; Taub, 1962) and via variational principles (Chandrasekhar, 1964; Cocke, 1965; Harrison *et al.*, 1965). The equilibrium equations as well as the pulsation equations were machine-integrated, with the eigenfunctions being matched to the surface boundary condition via the "shooting method" described in the Appendix.

2. Carbon Dwarf

The model treated in this section consists of pure C^{12}. A model with such a composition is stable against electron capture at $\log \varrho = 10.5$ g/cm^3 since the Fermi level at that density is 12.4 MeV, which is well below the point of instability against electron capture at 13.4 MeV. In Table I results are given for models with degenerate electron pressure only, while Table II gives the results corrected for coulomb, Thomas-Fermi, and exchange interactions. The relative magnitude of these corrections to the degener-

TABLE I

General relativistic white dwarf models with degenerate electron pressure only

Composition: carbon			
cgs units			
Log density	Mass/10^{33}	Radius/10^{8}	Fundamental period
10.5	2.78	.934	unstable
10	2.77	1.31	2.22
9.5	2.74	1.81	2.26
9	2.67	2.46	2.71
8.5	2.53	3.28	3.44
8	2.31	4.30	4.51
7.5	1.99	5.55	6.13
7	1.59	7.04	8.68
6.5	1.17	8.81	13.0
6	.784	10.9	20.5

TABLE II

General relativistic white dwarf models with coulomb, Thomas-Fermi, and exchange interactions considered

Composition: carbon			
cgs units			
Log density	Mass/10^{33}	Radius/10^{8}	Fundamental period
10.5	2.72	.926	unstable
10	2.72	1.30	2.19
9.5	2.68	1.79	2.24
9	2.61	2.43	2.69
8.5	2.47	3.24	3.41
8	2.25	4.24	4.46
7.5	1.94	5.45	6.04
7	1.55	6.89	8.53
6.5	1.12	8.56	12.7
6	.743	10.5	19.9

ate electron equation of state are discussed by Salpeter (1961) who gives his results in tabular as well as analytic form.

The pressure associated with the zero-point energy of the ions causes a further reduction in the stellar pulsation period; unlike the degenerate electrons, the ions are not relativistic. The expression for the ion pressure can be obtained from the zero-point energy via the well-known relation $p = -\mathrm{d}E/\mathrm{d}V$ (E being the energy/particle and V the volume/particle) which reduces to

$$p = -(4\pi a_0^3 r_e^2)^{-1}\, \mathrm{d}E/\mathrm{d}r_e \tag{4}$$

where a_0 is the Bohr radius and r_e is a dimensionless parameter used by Salpeter (1961).

This expression agrees with that given by Salpeter (1961) once a misprint in his paper is corrected. In terms of the relativity parameter x (the ratio of electron Fermi momentum to mc), the expression for the pressure takes the convenient form

$$p = (x^4/9\pi^2)\,(mc/\hbar)^3\,\mathrm{d}E/\mathrm{d}x \tag{5}$$

Table III gives the results corrected for ion zero-point energy as well as the above-mentioned interactions.

It may be interesting to note that at a density of $10^{10.5}$ g/cm^3, the carbon white dwarf models in Tables I, II, III are dynamically unstable due to general relativity, while in Newtonian mechanics the corresponding models are dynamically stable (Cohen, 1968; Skilling, 1968a; Faulkner and Gribbin, 1968). Thus, general relativity makes a qualitative difference in the results.

3. Inhomogeneous Models

In this section, we treat white dwarf models with inhomogeneous composition by integrating Einstein's equilibrium equations and the equations for radial pulsations about the equilibrium positions. Since m, φ, and p are continuous, the Equations (1) to (3) of the preceding section can be integrated across discontinuities in composition (and density) with no special treatment of these surfaces being necessary. A similar procedure was used with the pulsation equations since the eigenfunction ξ and y_2 (see appendix) are continuous. To minimize errors, none of the above quantities was allowed to change by more than 5% in any integration step; such a procedure gives rise to a large number of zones in the outer regions of the star where the pressure is changing most rapidly. The pressure changes by many orders of magnitude while the radius remains almost constant near the surface. When the trial eigenvalue is sufficiently close to the actual eigenvalue, the 'eigenfunction' ξ often remains almost constant from center to surface. If the trial eigenvalue is not close, however, the trial eigenfunction may change rapidly and become quite large near the surface of the star. The computational method is described more fully in the appendix.

TABLE III

General relativistic white dwarf models with coulomb, Thomas-Fermi and exchange interaction, as well as ion zero point motion considered

Composition: carbon			
cgs units			
Log density	Mass/10^{33}	Radius/10^8	Fundamental period
10.5	2.72	.926	unstable
10	2.72	1.30	2.15
9.5	2.68	1.79	2.23
9	2.61	2.43	2.69
8.5	2.47	3.24	3.41
8	2.25	4.24	4.46

The model treated here consists of a carbon outer envelope with a core composed of carbon-burning products. The core was assumed to have the following composition (Arnett and Truran, 1968) (by relative numbers of nuclei): 1% of $A=16$, 6% of $A=23$, 41% of $A=20$, and 52% of $A=24$. These percentages are based on preliminary results of Arnett and Truran (1968) for carbon-burning at 10^9 K. The charge on these nuclei depends on the Fermi level of the degenerate electrons. At high densities, the electron Fermi level becomes so high that (1) nuclei which are normally unstable against beta decay become stable (since the decay electrons have insufficient energy to occupy the available states above the Fermi level) and (2) nuclei which are normally stable against beta decay become unstable because of electron capture (since some degenerate electrons have sufficient energy to make electron capture energetically possible and the product stable). Such a model may be formed by stellar evolution (Arnett and Truran, 1968).

The carbon-burning products which are normally stable against beta decay and electron capture are O^{16}, Ne^{20}, Na^{23}, and Mg^{24}. With increasing density (and Fermi level), these nuclei capture electrons, changing in the table below toward the right into the indicated nuclei at the indicated electron capture thresholds (in MeV) (Garvey and Kelson, 1966):

O^{16}	10.4	C^{16}				
Ne^{20}	7.03	O^{20}	21.22	C^{20}		
Na^{23}	4.4	Ne^{23}	11.15	F^{23}	13.87	O^{23}
Mg^{24}	5.52	Ne^{24}	15.91	O^{24}		

This tabulation is not valid for Fermi levels above 23.33 MeV. Nuclei with even mass numbers undergo two successive electron captures, since the thresholds next following those listed are lower than the ones listed.

The change in Z for a given A was assumed to occur at the electron capture threshold. The partial pressures (from the Hamada-Salpeter (1961) corrections), for each A and Z, were added to give the total correction to the pressure. The energy density was corrected in a similar manner. In view of the crudeness of the corrections themselves, further refinements in the method of combining the corrections will not be considered here. The outer carbon envelope was assumed in some models to begin at a mass of 2×10^{33} g, and in other models at 10^{33} g.

Because the particle half-lives are much larger than the pulsation periods ($\sim$1 sec), we assumed a fixed composition when calculating the adiabatic index γ. This assumption complicates the calculation of γ since the static and dynamic equations of state are not the same; the static equation of state contains a large contribution from electron capture and beta decay whereas the dynamic equation of state does not. Because the two equations of state differ, the star can pulsate with a non-infinite (real) pulsation period beyond the mass peak. In that region, the star can also exhibit an imaginary pulsation period corresponding to slow collapse as electron capture takes place. Detailed discussion of electron capture and beta decay during stellar pulsations will be given elsewhere by Chiu and Cohen.

Table IV, V, VI, and VII show that the peak of the mass vs central density curve does not correspond to the point of relativistic instability. Models with maximum mass and with minimum period are marked with asterisks. These models become unstable due to electron capture as would be the case for Newtonian mechanics. Electron capture in white dwarfs has been considered, e.g., by Baglin (1966, 1968) and Hansen and Wheeler (1969).

4. Discussion

From Tables I and II it can be seen that white dwarf models which take into account coulomb, Thomas-Fermi and exchange interactions have a lower fundamental pulsation period and mass than models with the same central density but with degenerate

TABLE IV

Relativistic white dwarf models with degenerate electron pressure only

Composition: core of mass $M_\odot$ composed of carbon-burning products with a carbon envelope			
cgs units			
Log density	Mass/10^{33}	Radius/10^8	Fundamental period
11	1.964	.796	unstable
10.7	2.06	.954	unstable
10.6	2.11	1.03	3.89
10.5	2.16	1.11	2.55
10.2	2.36	1.34	2.04*
10	2.52	1.49	2.07
9.5	2.72*	1.82	2.26
9	2.65	2.46	2.71
8.5	2.51	2.28	3.44
8	2.29	4.30	4.51

TABLE V

Relativistic white dwarf models with coulomb, Thomas-Fermi, and exchange interactions as well as ion zero point motion considered

Composition: core of mass $M_\odot$ composed of carbon-burning products with a carbon envelope			
cgs units			
Log density	Mass/10^{33}	Radius/10^8	Fundamental period
11	1.91	.783	unstable
10.7	2.00	.939	unstable
10.6	2.05	1.02	3.21
10.5	2.10	1.09	2.36
10.2	2.29	1.33	1.98*
10	2.44	1.48	2.02
9.5	2.63	1.80	2.22
9	2.57*	2.43	2.68
8.5	2.43	3.23	3.40
8	2.21	4.22	4.44

TABLE VI

Relativistic white dwarf models with degenerate electron pressure only

Composition: core of mass $M_\odot/2$ composed of carbon-burning products with a carbon envelope			
	cgs units		
Log density	Mass/10^{33}	Radius/10^8	Fundamental period
11	2.16	.855	unstable
10.7	2.19	.988	unstable
10.6	2.20	1.06	3.61
10.5	2.23	1.12	2.54
10.2	2.37	1.35	2.05*
10.0	2.52	1.49	2.07
9.5	2.72*	1.82	2.26
9.0	2.66	2.46	2.71
8.5	2.52	3.28	3.44
8.0	2.30	4.31	4.52

TABLE VII

Relativistic white dwarf models with coulomb, Thomas-Fermi, and exchange interactions as well as ion zero point motion considered

Composition: core of mass $M_\odot/2$ composed of carbon-burning products with a carbon envelope			
	cgs units		
Log density	Mass/10^{33}	Radius/10^8	Fundamental period
11	2.10	.849	unstable
10.7	2.12	.982	unstable
10.6	2.14	1.05	3.00
10.5	2.17	1.12	2.34
10.2	2.31	1.34	1.99*
10	2.46	1.48	2.03
9.5	2.65*	1.81	2.23
9	2.58	2.44	2.69
8.5	2.45	3.25	3.41
8	2.26	4.25	4.46

electron pressure only. The ion zero-point energy increases with density relative to the above mentioned interactions, and manifests itself at high densities through an additional reduction of the fundamental pulsation frequency as can be seen from comparison of Tables II and III. The corrected and uncorrected models are compared graphically in Figures 1 and 2. From these graphs it can be noted that models with electron pressure only are unstable at $\log\varrho = 10.3$, while corrected models are stable at this density. Thus the interactions delay the onset of general relativistic instability. (This is not surprising, of course, since models which consider these interactions have a lower fundamental pulsation period than models which do not.)

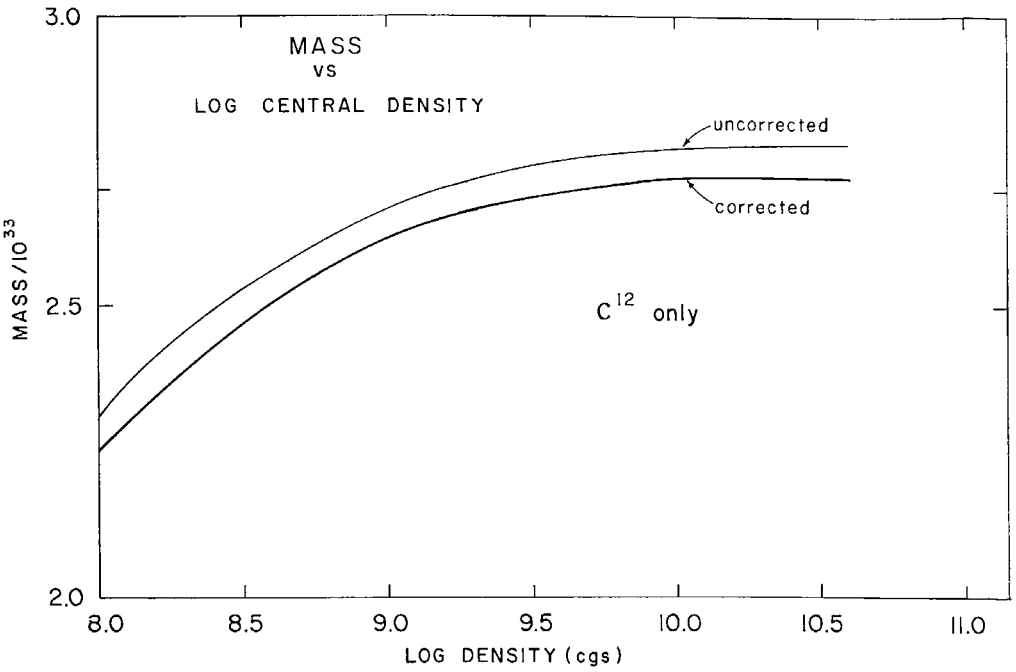

Fig. 1. Mass of a white dwarf composed of pure C^{12} as a function of central density.

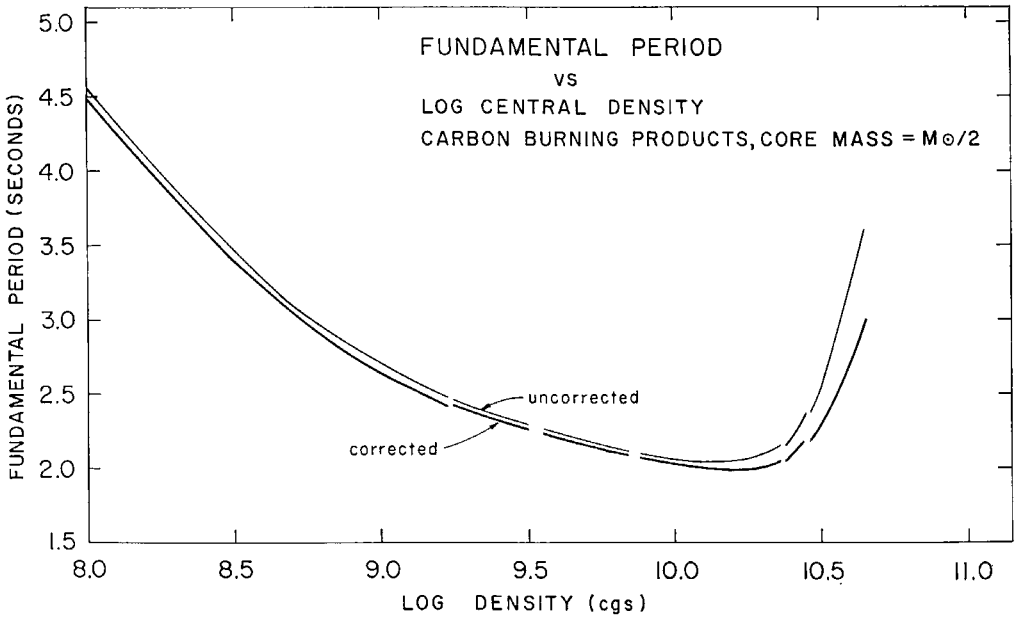

Fig. 2. Fundamental period of a white dwarf composed of pure C^{12} as a function of central density.

Tables I–III and Figures 1 and 2 indicate that dynamical instability sets in near the peak of the mass curve. These results are in agreement with theoretical expectations (Oppenheimer and Volkoff, 1939; Harrison *et al.*, 1965; Cocke, 1965). Of course, from the results of numerical integration of equations, one can only check approximately this expectation.

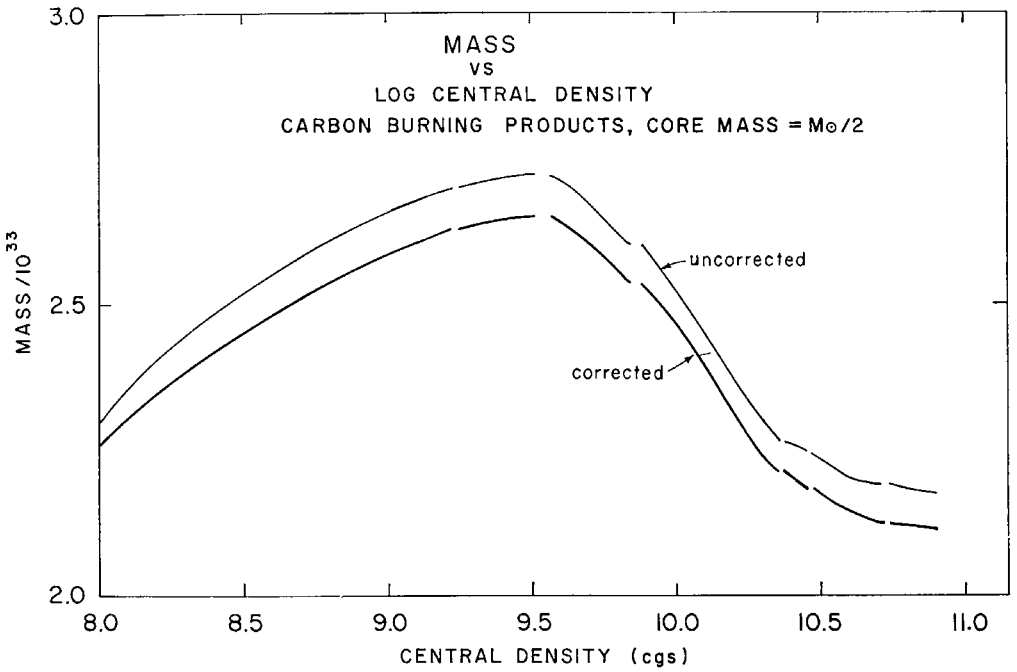

Fig. 3. Mass of a white dwarf with a carbon-burning products core and a C^{12} envelope as a function of central density.

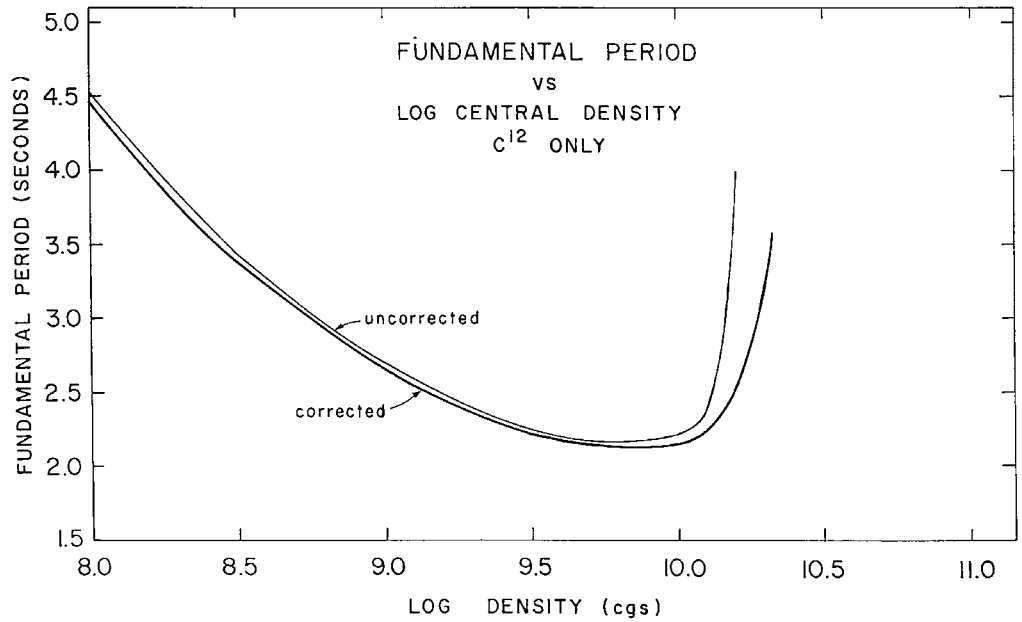

Fig. 4. Fundamental period of a white dwarf with a carbon-burning products core and a C^{12} envelope as a function of central density.

The results for models composed of a carbon-burning product core and a C^{12} envelope are displayed in Tables 4–7 and graphically in Figures 3–6. As with the pure carbon models, the interactions alter the equation of state in such a way that the corrected models have a slightly lower mass and shorter fundamental pulsation period than models with the same central density but a free electron equation of state.

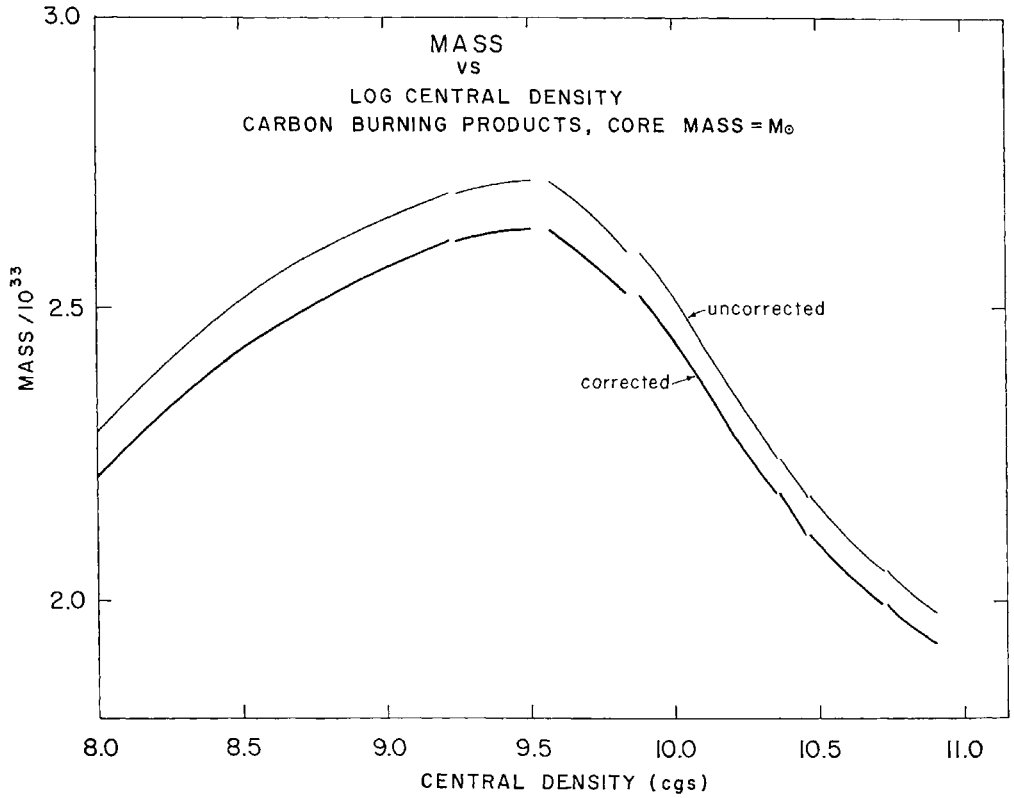

Fig. 5. Mass of a white dwarf with a carbon-burning products core and a C^{12} envelope as a function of central density.

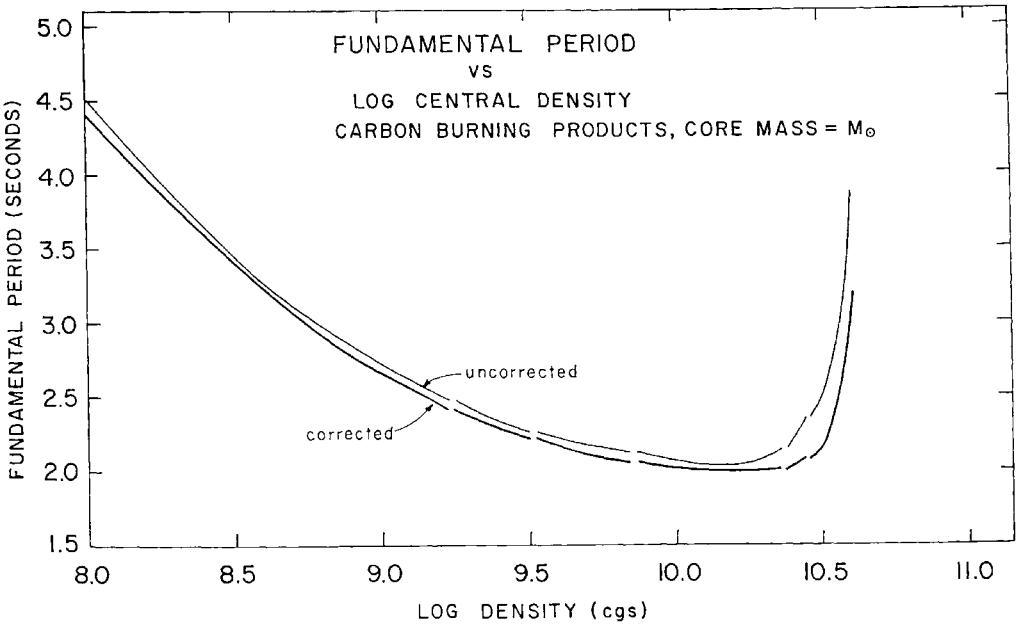

Fig. 6. Fundamental period of a white dwarf with a carbon-burning products core and a C^{12} envelope as a function of central density.

However, unlike the pure carbon models, the carbon-burning product models have a mass curve (Figures 3 and 5) whose peak does *not* correspond to the point of dynamical instability, Figures 4 and 6. This is because the carbon-burning product mass curve exhibits a peak due to electron capture while the peak of the pure carbon mass curve is due to general relativistic effects.

Note in Figures 3–6 that there are discontinuities in the central density which correspond to the electron capture thresholds. The masses and vibration periods are continuous, and hence there are horizontal displacements of the curves at the threshold discontinuities.

The fundamental periods of these more realistic white dwarf models appear to be too large to be of interest for pulsar models.

Acknowledgements

For helpful discussion we are indebted to W. J. Cocke, M. Halem, E. E. Salpeter, and A. Schindler. We are grateful to K. S. Thorne for a critical reading of the manuscript and for several helpful suggestions regarding the presentation. We should also like to thank Arnold Cohen for his help in preparing the table and graphs. This paper is based, in part, on an invited paper by J. M. Cohen, Pulsar Conference, 20–21 May, New York. The work of one of us (J. M. C.) was supported, in part, by an NAS–NRC Research associateship sponsored by the National Aeronautics and Space Administration. Partial support was provided by the National Science Foundation and by the National Aeronautics and Space Administration.

Appendix

COMPUTATIONAL METHOD

This section first explains the main procedures for the integration of the equilibrium and pulsation equations. Then, some of the details connected with the iterative solution of the eigenvalue problem are described.

Equations (1), (2), (3) are integrated using a standard Runga-Kutta method as found, for example, in Freiberger (1960). The right hand sides of (1), (2), (3) have finite jumps because the density ϱ jumps at electron capture thresholds. At such places the Runga-Kutta iteration cannot be fourth-order accurate because higher derivatives than the first do not exist. An error of the form const$\cdot \Delta r$ is made at each jump, and since the number of jumps is less than one percent of the number of zones, the overall error is no worse than the overall error which would be obtained from integration methods which have errors of the form const$\cdot (\Delta r)^2$. (It is possible to make the error near the contact discontinuities arbitrarily small by approaching the jump in small steps. Such a method was tried and gave good answers, but the results reported here use the simpler method of computing, which gave results that agreed with the more complicated method to better than 1 part in 2000.)

The solutions so obtained are subsequently used to determine the coefficients of the

pulsation Equation (6).

$$y_1' = -(Q + \omega^2 W)\, y_2$$
$$y_2' = y_1/P \tag{6}$$

where primes denote differentiation with respect to r,

$$P = \mathrm{e}^{3\varphi + \lambda/2}\gamma p/r^2$$
$$W = \mathrm{e}^{\varphi + 3\lambda/2}(\varrho + p/c^2)/r^2$$
$$Q = -4p'/r - 8\pi(\varrho + p/c^2)\, p\, \mathrm{e}^{\lambda}\, G/c^2 + (p')^2/(\varrho c^2 + p)\, \frac{\mathrm{e}^{3\varphi + \lambda/2}}{p^2}$$
$$Y_2 = r^3\, \mathrm{e}^{-\varphi}\, \xi$$
$$e^{\lambda} = 1/(1 - 2\,Gm/rc^2)$$
$$\gamma = (\varrho/p + 1/c^2)\left.\frac{\partial p}{\partial \varrho}\right|_{\text{const. entropy}}$$

ξ is the relative displacement $(\delta r/r)$.

ω is the pulsation frequency,

and

y_1 is an auxiliary variable introduced to change the second order pulsation equation into a pair of first order equations.

The eigenvalues are then found via the 'shooting method'.

The trial eigenvalue $X = \omega^2$ is systematically varied until the solution $(y_1(X), y_2(X))$ fits an outer boundary condition given by

$$y_1 = p y_2 \cdot (4\,\mathrm{e}^{2\varphi} + \omega^2 r^3/Gm + Gm/rc^2)/r^3\,.$$

All the trial eigensolutions have a boundary condition at the center given by

$$y_1 = 3\gamma p\, \mathrm{e}^{3\varphi}$$
$$y_2 = r^3\, \mathrm{e}^{\varphi}$$

The problem then is to find a root of

$$f(X) = y_1(X) - p y_2(X)\,(4\,\mathrm{e}^{2\varphi} + Xr^3/Gm + Gm/rc^2)/r^3 = 0$$

at the surface.

Because each evaluation of $f(X)$ requires integrating a system of differential equations, it is tempting to try using the swiftly converging Newton's method (8)

$$X_{n+1} = X_n - f(X_n)/f'(X_n)$$

to find the roots of (7), although an expression for $f'(X)$ is not readily available. A numerical approximation to $f'(X)$ was made and the procedure was tried but did not always converge. The Equation (7) is actually solved using linear interpolation which always converged in fewer than 20 steps.

For the determination of the eigenvalue and eigenfunctions, this method is easier to use than that of Bardeen *et al.* (1966) since it involves only integrating out from the center of the star with various trial eigenvalues until the Condition (7) is satisfied to the accuracy desired. Unless the trial eigenvalue is very close to the correct value, the eigenfunction becomes large near the surface. For all of the calculations performed in connection with this paper, the eigenfunctions were well behaved near the surface when the eigenvalues were sufficiently close to the correct value.

Bardeen *et al.* (1966) obtained their eigenfunctions by integrating out from the center and in from the surface and joining in the interior of the star. Both of these methods are 'shooting methods' in that one starts at one (or several) point and varies a parameter (the trial eigenvalue) until one satisfies a condition at another point. As mentioned earlier a feature of this program is that the dependent variables chosen allow accurate integration through points of density discontinuity without requiring special treatment at these points.

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